Local and global extrema 2

Identify all the local maxima and minima (if they exist) for the given function, and ascertain if each local extremum qualifies as a global extremum.

$$f(x, y, z) = 1 - x^2 - y^2 - z^2$$

Solution

First-order conditions:

$$\begin{aligned} f_1(x,y,z) &= -2x = 0, \\ f_2(x,y,z) &= -2y = 0, \\ f_3(x,y,z) &= -2z = 0. \end{aligned}$$

There is a unique solution, $(x,y,z) = (0,0,0)$. The Hessian matrix is

$$\begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

We can verify the conditions for minima and maxima by examining whether the Hessian matrix is positive definite or negative definite. A matrix is positive definite if all of its leading principal minors are positive. Conversely, a matrix is negative definite if the signs of the leading principal minors alternate, beginning with a negative.

The leading principal minors are

$$D_1 = -2 < 0$$

$$D_2 = 4 > 0$$

$$D_3 = -2 \cdot 4 = -8 < 0$$

So that $(x_1, x_2, x_3) = (0,0,0)$ is a local maximizer; the value at this maximum is 1. The Hessian is negative definite for all (x,y,z), so that f is concave. Thus $(0,0,0)$ is a global maximizer.